

A toy model of bosonic non-canonical quantum field

Marek Czachor and Monika Syty

Katedra Fizyki Teoretycznej i Metod Matematycznych

Politechnika Gdańska, ul. Narutowicza 11/12, 80-952 Gdańsk, Poland

Abstract

A harmonic oscillator is an indefinite-frequency one if the parameter ω is replaced by an operator. An ensemble of N such oscillators may be regarded as a toy model of a bosonic quantum field. All the possible frequencies associated with a given problem are present already in a single oscillator and N can be finite. Due to the operator character of ω the resulting algebra of creation-annihilation operators is non-canonical. In the limit of large N one recovers perturbation theory formulas of the canonical quantum field theory but with form factors automatically built in. Vacuum energy of the ensemble is finite, a fact discussed in the context of the cosmological constant problem. Space of states is given by a vector bundle with Fock-type fibers. Interactions of the field with 2-level systems, including Rabi oscillations and spontaneous emission, are discussed in detail.

I. INTRODUCTION

It is well known that standard canonical procedures of field quantization result in various infinities which have to be removed in a more or less ad hoc manner. In spite of unquestionable successes of quantum field theories, there exists a possibility that we are still overlooking an ingredient which is essential for a physically consistent field quantization.

Quite recently it was pointed out [1] that the very first step of field quantization may be

performed incorrectly. The point is that in the description of quantum harmonic oscillators *à la* Heisenberg one treats the frequency ω in

$$H_\omega = \frac{\hat{p}^2}{2m} + \frac{m\omega^2 \hat{q}^2}{2} \quad (1)$$

as a parameter. A mathematical implication of this fact is the algebra of canonical commutation relations (CCR) characterizing creation and annihilation operators.

However, thinking of real systems (such as a simple pendulum, or a string) one realizes that ω typically depends on other physical quantities (say, positions) which in quantum theories are represented by operators. As a consequence it is not entirely irrelevant to ask what would be changed in the theory if one quantized not only p and q but also ω [2].

The main physical conclusion of the analysis given in [1] is that at least vacuum and ultraviolet infinities may disappear as a result of this single modification [3]. Mathematically the effect is rooted in non-canonical commutation relations (non-CCR) which naturally replace CCR.

The purpose of the present paper is to take a closer look at the meaning of the main assumption of [1]. We return to the first step of the construction and concentrate on a nonrelativistic harmonic oscillator. We replace ω by an operator. We discuss in detail creation and annihilation operators and modifications of CCR. We arrive at the same non-CCR algebra as in [1] but now in a slightly modified representation. Evaluating averages of position and momentum we arrive at expressions resembling Fourier expansions of classical fields.

The next step is to discuss interactions of such indefinite-frequency oscillators with two-level systems. We study the excited-state survival amplitude for a two-level system interacting with N oscillators and obtain an exact formula in terms of Laplace transforms.

For any N and ω with only one frequency we obtain the Rabi oscillation. A more realistic case is when the set of all the ω s corresponds to a cavity spectrum. One expects Rabi oscillations also if one frequency is in resonance with the two-level system while the remaining ω s are very far from resonance. For small N an appropriate probability differs

essentially from the Jaynes-Cummings solution. However, we show that for large N one expects a solution analogous to the standard one.

We next show that to any order of perturbation theory and to any given precision the perturbative predictions of the canonical theory may be reconstructed in the non-canonical formalism with a *finite* number of oscillators, and the “canonical infinities” do not occur. We thus generalize the results from [1] which were shown explicitly only in low orders of perturbation theory.

Since in the non-canonical formalism the main feature is an automatic elimination of divergent expressions, it is interesting to discuss implications of finite vacuum contribution to the cosmological constant problem. We show that in the simplest approach the well known formula for the vacuum energy density, $\rho_{\text{old}} \approx \frac{\hbar}{16\pi^2 c^3} \omega_{\text{max}}^4$ is replaced by $\rho_{\text{new}} \approx \frac{3}{8} \hbar \omega_{\text{max}} \frac{\langle N \rangle}{V}$ where $\langle N \rangle/V$ is the average number of oscillators in volume V . The physical meaning of the cut-off frequency is also different from the usual one: This is not the cut-off in the energy spectrum, but the frequency above which the vacuum probability density of ω ’s is negligible. The very existence of such a parameter in non-canonical theories trivially follows from square integrability of vacuum wave functions. Experimental values of vacuum energy density allow to estimate the average number of oscillators in a unit volume. The resulting number seems too small to guarantee consistency of non-canonical quantum optics with atomic measurements, which suggests that before any comparison of non-canonical vacuum contributions with cosmology one should first discuss non-canonical theories with broken supersymmetry.

It should be stressed that what we are doing in the paper is not, strictly speaking, a new field quantization. We simply apply orthodox quantum mechanics to systems of many indefinite-frequency harmonic oscillators. The point we advocate is that for *any* number N of such oscillators (also $N < \infty$) the resulting ensemble may be regarded as a model for a spin-zero quantum field. In the limit $N \rightarrow \infty$ the ensemble has properties of the standard canonical quantum field, but in a version with form-factors automatically built in. Quantum field theoretic interpretation of the ensemble requires an appropriate construction

of the space of states. We show that the non-CCR state space is not a Fock one, but rather a vector bundle with the set of vacua in the role of a base space and Fock spaces playing the role of fibers.

II. INDEFINITE-FREQUENCY OSCILLATOR

We begin with (1) where $[\hat{q}, \hat{p}] = i\hbar\hat{1}$ and ω is a parameter. Wavefunctions related to H_ω are, in position representation, functions of q i.e. $\psi = \psi(q)$ with the normalization $\int_{-\infty}^{\infty} dq |\psi(q)|^2 = 1$. Now regard ω as a *quantum number*, i.e. an eigenvalue of some operator Ω . This means that the wavefunctions depend on two parameters, $\psi = \psi(\omega, q)$, and are normalized by

$$\int_{-\infty}^{\infty} dq \int_0^{\infty} d\omega |\psi(\omega, q)|^2 = 1,$$

if spectrum of Ω is continuous, or by

$$\sum_{\omega} \int_{-\infty}^{\infty} dq |\psi(\omega, q)|^2 = 1,$$

if spectrum is discrete [2].

A. Algebra of indefinite-frequency operators

At the level of observables we can formalize this by means of the operators

$$P = \hat{1} \otimes \hat{p} \tag{2}$$

$$Q = \hat{1} \otimes \hat{q} \tag{3}$$

$$\Omega = \hat{\omega} \otimes \hat{1} \tag{4}$$

$$H = \frac{P^2}{2m} + \frac{m\Omega^2 Q^2}{2} \tag{5}$$

whose explicit representation is

$$P\psi(\omega, q) = -i\hbar \frac{\partial \psi(\omega, q)}{\partial q} \tag{6}$$

$$Q\psi(\omega, q) = q\psi(\omega, q) \tag{7}$$

$$\Omega\psi(\omega, q) = \omega\psi(\omega, q) \quad (8)$$

$$H\psi(\omega, q) = \left(-\frac{\hbar^2}{2m} \frac{\partial^2}{\partial q^2} + \frac{m\omega^2 q^2}{2} \right) \psi(\omega, q). \quad (9)$$

$[Q, P] = i\hbar I$ where $I = \hat{1} \otimes \hat{1}$ [4]. As we can see no drastic changes were made so far by the reinterpretation of ω .

The creation and annihilation operators are defined via an obvious generalization of the standard formulas

$$a_\Omega = \sqrt{\frac{m\Omega}{2\hbar}} Q + i\sqrt{\frac{1}{2m\hbar\Omega}} P \quad (10)$$

$$a_\Omega^\dagger = \sqrt{\frac{m\Omega}{2\hbar}} Q - i\sqrt{\frac{1}{2m\hbar\Omega}} P \quad (11)$$

$$I = [a_\Omega, a_\Omega^\dagger] \quad (12)$$

and satisfy

$$H = \frac{\hbar\Omega}{2} (a_\Omega^\dagger a_\Omega + a_\Omega a_\Omega^\dagger) \quad (13)$$

Assume for simplicity that spectrum of Ω is discrete, i.e. its spectral representation reads

$$\Omega = \sum_\omega \omega |\omega\rangle \langle \omega| \otimes \hat{1}. \quad (14)$$

The Hamiltonian can be now written as

$$H = \sum_\omega \frac{\hbar\omega}{2} (a_\omega^\dagger a_\omega + a_\omega a_\omega^\dagger) \quad (15)$$

where

$$a_\omega = \sqrt{\frac{m\omega}{2\hbar}} |\omega\rangle \langle \omega| \otimes \hat{q} + i\sqrt{\frac{1}{2m\hbar\omega}} |\omega\rangle \langle \omega| \otimes \hat{p} \quad (16)$$

$$= |\omega\rangle \langle \omega| \otimes \left(\sqrt{\frac{m\omega}{2\hbar}} \hat{q} + i\sqrt{\frac{1}{2m\hbar\omega}} \hat{p} \right) \quad (17)$$

$$= |\omega\rangle \langle \omega| \otimes \hat{a}_\omega. \quad (18)$$

Here \hat{a}_ω is the standard annihilation operator for an oscillator with frequency given by the *parameter* ω . \hat{a}_ω satisfies canonical commutation relations (CCR)

$$[\hat{a}_\omega, \hat{a}_\omega^\dagger] = \hat{1}. \quad (19)$$

In the standard formalism one does not ask for the commutator $[\hat{a}_\omega, \hat{a}_{\omega'}^\dagger]$ since a single oscillator has only one frequency parameter and the question is physically ill posed. Two different ω and ω' can occur only if one has two different oscillators and then the commutator vanishes.

The algebra of a_ω is non-canonical (non-CCR):

$$[a_\omega, a_{\omega'}^\dagger] = \delta_{\omega\omega'} |\omega\rangle\langle\omega| \otimes \hat{1} =: \delta_{\omega\omega'} I_\omega. \quad (20)$$

Now the commutator corresponds to a well posed physical problem.

A notable property of the non-CCR algebra (20) is the resolution of identity satisfied by the right-hand-side of the commutator:

$$\sum_{\omega} I_\omega = I. \quad (21)$$

B. Indefinite-frequency states

Eq. (9) shows that energy eigenvector corresponding to the eigenvalue

$$E(\omega, n) = \hbar\omega(n + \tfrac{1}{2}) \quad (22)$$

is

$$|\omega, n\rangle = |\omega\rangle |n_\omega\rangle. \quad (23)$$

By $|n_\omega\rangle$ we denote the basis associated with the decomposition

$$\hat{a}_\omega = \sum_{n=0}^{\infty} \sqrt{n+1} |n_\omega\rangle\langle n+1_\omega| \quad (24)$$

of the CCR annihilation operator. The associated position-space wavefunction is

$$\langle q | n_\omega \rangle \sim e^{-q^2 m \omega / (2\hbar)} h_n(q \sqrt{m \omega / (2\hbar)}) \quad (25)$$

with $h_n(\cdot)$ the Hermite polynomial.

One of the properties that make non-CCR oscillator interesting is that the average energy evaluated in the state

$$|\psi\rangle = \sum_{\omega, n} \psi(\omega, n) |\omega, n\rangle \quad (26)$$

can be written as

$$\langle\psi|H|\psi\rangle = \sum_{\omega, n} \hbar\omega(n + \frac{1}{2}) |\psi(\omega, n)|^2 \quad (27)$$

i.e. in a form which looks like an average energy of an ensemble of independent oscillators with different frequencies: A *single* indefinite-frequency oscillator in many respects resembles an *ensemble* of many independent oscillators.

We define a vacuum state as *any* state $|O\rangle$ which is annihilated by a_Ω i.e.

$$|O\rangle = \sum_{\omega} O(\omega) |\omega, 0\rangle. \quad (28)$$

Let us note that a general vacuum state can be time dependent. In particular

$$|O_t\rangle = e^{i\Omega t/2} |O\rangle \quad (29)$$

is also a vacuum state. The unitary transformation (29) defines a “vacuum picture” whose dynamics is given by the Hamiltonian

$$\tilde{H} = \hbar\Omega a_\Omega^\dagger a_\Omega \quad (30)$$

Having a vacuum we can define N th excited states by

$$|N\rangle = \frac{1}{\sqrt{N!}} (a_\Omega^\dagger)^N |O\rangle. \quad (31)$$

$|N\rangle$ represents a superposition of oscillators with different frequencies but the same level of excitation.

Coherent states can be defined in the usual way via the displacement operator

$$D(z) = \exp(z a_{\Omega}^{\dagger} - \bar{z} a_{\Omega}) \quad (32)$$

$$|z\rangle = D(z)|O\rangle \quad (33)$$

$$a_{\Omega}|z\rangle = z|z\rangle \quad (34)$$

One can further generalize coherent states by taking any operator function

$$f_{\Omega} = \sum_{\omega} f(\omega)|\omega\rangle\langle\omega| \otimes \hat{1}. \quad (35)$$

and the displacement operator

$$D(f_{\Omega}) = \exp(f_{\Omega} a_{\Omega}^{\dagger} - f_{\Omega}^{\dagger} a_{\Omega}) \quad (36)$$

$$|f_{\Omega}\rangle = D(f_{\Omega})|O\rangle \quad (37)$$

$$= \sum_{\omega} O(\omega) e^{f(\omega) a_{\omega}^{\dagger} - \overline{f(\omega)} a_{\omega}} |\omega, 0\rangle \quad (38)$$

$$= \sum_{\omega} O(\omega) |\omega\rangle e^{f(\omega) \hat{a}_{\omega}^{\dagger} - \overline{f(\omega)} \hat{a}_{\omega}} |0_{\omega}\rangle \quad (39)$$

$$= \sum_{\omega} O(\omega) |\omega\rangle |f(\omega)_{\omega}\rangle \quad (40)$$

$$a_{\Omega}|f_{\Omega}\rangle = f_{\Omega}|f_{\Omega}\rangle = \sum_{\omega} O(\omega) f(\omega) |\omega\rangle |f(\omega)_{\omega}\rangle \quad (41)$$

Eq. (41) means that coherent states are generalized eigenstates of annihilation operators.

C. Single-oscillator “fields”

The Heisenberg picture dynamics is given by the familiar formulas

$$e^{iHt/\hbar} a_{\Omega} e^{-iHt/\hbar} = e^{-i\Omega t} a_{\Omega} = e^{i\tilde{H}t/\hbar} a_{\Omega} e^{-i\tilde{H}t/\hbar}, \quad (42)$$

$$e^{iHt/\hbar} a_{\omega} e^{-iHt/\hbar} = e^{-i\omega t} a_{\omega} = e^{i\tilde{H}t/\hbar} a_{\omega} e^{-i\tilde{H}t/\hbar} \quad (43)$$

implying

$$Q_t = e^{iHt/\hbar} Q e^{-iHt/\hbar} = \sqrt{\frac{\hbar}{2\Omega m}} (e^{-i\Omega t} a_{\Omega} + e^{i\Omega t} a_{\Omega}^{\dagger}) \quad (44)$$

$$P_t = e^{iHt/\hbar} P e^{-iHt/\hbar} = -i\sqrt{\frac{\hbar\Omega m}{2}} (e^{-i\Omega t} a_{\Omega} - e^{i\Omega t} a_{\Omega}^{\dagger}) \quad (45)$$

Evaluating the coherent-state averages

$$\langle f_\Omega | Q_t | f_\Omega \rangle = \sum_\omega |O(\omega)|^2 \sqrt{\frac{\hbar}{2\omega m}} \left(e^{-i\omega t} f(\omega) + e^{i\omega t} \overline{f(\omega)} \right) \quad (46)$$

$$\langle f_\Omega | P_t | f_\Omega \rangle = -i \sum_\omega |O(\omega)|^2 \sqrt{\frac{\hbar \omega m}{2}} \left(e^{-i\omega t} f(\omega) - e^{i\omega t} \overline{f(\omega)} \right) \quad (47)$$

we realize that a single harmonic oscillator with indefinite frequency is an object closely related to quantum fields.

III. MANY OSCILLATORS

An ensemble of many indefinite-frequency oscillators has properties analogous to a quantum field. Formally the multi-oscillator structure is constructed as follows.

Let A be an operator at a “single-oscillator” level. If \mathcal{H} is the Hilbert space of one-oscillator states then $A : \mathcal{H} \rightarrow \mathcal{H}$. Let

$$\underline{\mathcal{H}} = \oplus_{n=1}^{\infty} \otimes_s^n \mathcal{H} \quad (48)$$

be the Hilbert space of states corresponding to an indefinite number of oscillators; $\otimes_s^n \mathcal{H}$ stands for a space of symmetric states in $\underbrace{\mathcal{H} \otimes \dots \otimes \mathcal{H}}_n$. We introduce the following notation for operators defined at the multi-oscillator level:

$$\oplus_{\alpha_n} A = \alpha_1 A \oplus \alpha_2 (A \otimes I + I \otimes A) \oplus \alpha_3 (A \otimes I \otimes I + I \otimes A \otimes I + I \otimes I \otimes A) \oplus \dots \quad (49)$$

Here $\oplus_{\alpha_n} A : \underline{\mathcal{H}} \rightarrow \underline{\mathcal{H}}$ and α_n are real or complex parameters.

The following properties follow directly from the definition

$$[\oplus_{\alpha_n} A, \oplus_{\beta_n} B] = \oplus_{\alpha_n \beta_n} [A, B] \quad (50)$$

$$e^{\oplus_{\alpha_n} A} = \oplus_{n=1}^{\infty} \underbrace{e^{\alpha_n A} \otimes \dots \otimes e^{\alpha_n A}}_n \quad (51)$$

$$e^{\oplus_1 A} \oplus_{\beta_n} B e^{-\oplus_1 A} = \oplus_{\beta_n} e^A B e^{-A} \quad (52)$$

Identity operators at \mathcal{H} and $\underline{\mathcal{H}}$ are related by

$$\underline{I} = \oplus_n I. \quad (53)$$

Define multi-oscillator creation and annihilation operators by

$$\underline{a}_\Omega = \oplus_{\frac{1}{\sqrt{n}}} a_\Omega, \quad (54)$$

$$\underline{a}_\Omega^\dagger = \oplus_{\frac{1}{\sqrt{n}}} a_\Omega^\dagger \quad (55)$$

$$\underline{a}_\omega = \oplus_{\frac{1}{\sqrt{n}}} a_\omega, \quad (56)$$

$$\underline{a}_\omega^\dagger = \oplus_{\frac{1}{\sqrt{n}}} a_\omega^\dagger. \quad (57)$$

Then

$$[\underline{a}_\Omega, \underline{a}_\Omega^\dagger] = \oplus_{\frac{1}{n}} [a_\Omega, a_\Omega^\dagger] = \oplus_{\frac{1}{n}} I = \underline{I} \quad (58)$$

$$[\underline{a}_\omega, \underline{a}_{\omega'}^\dagger] = \oplus_{\frac{1}{n}} [a_\omega, a_{\omega'}^\dagger] = \oplus_{\frac{1}{n}} \delta_{\omega\omega'} I_\omega =: \delta_{\omega\omega'} \underline{I}_\omega \quad (59)$$

$$(60)$$

The apparently artificial factor $1/\sqrt{n}$ in (54)–(57) is needed to maintain the CCR condition (58). Another consequence of this choice of α_n is the resolution of identity

$$\sum_{\omega} \underline{I}_\omega = \underline{I} \quad (61)$$

which is the multi-oscillator counterpart of (21). The Hamiltonian of an ensemble of noninteracting oscillators is

$$\underline{H} = \oplus_1 H \quad (62)$$

$$= \oplus_1 \frac{\hbar\Omega}{2} (a_\Omega a_\Omega^\dagger + a_\Omega^\dagger a_\Omega) \quad (63)$$

$$= \oplus_1 \left(\frac{P^2}{2m} + \frac{m\Omega^2 Q^2}{2} \right) \quad (64)$$

This is the standard form of Hamiltonian corresponding to many noninteracting particles.

Denoting $\underline{\Omega} = \oplus_1 \Omega$ and proceeding similarly to the single-oscillator case we can define the vacuum picture by means of the unitary transformation

$$|\underline{\tilde{\psi}}\rangle = e^{i\underline{\Omega}t/2} |\underline{\psi}\rangle \quad (65)$$

and the vacuum-picture Hamiltonian is

$$\underline{\tilde{H}} = \oplus_1 \tilde{H} = \oplus_1 \hbar \Omega a_\Omega^\dagger a_\Omega. \quad (66)$$

There is a subtle difference between the way we introduce the vacuum picture and the standard way of removing the infinite energy of vacuum. In our case the operation is given by a well defined unitary operator, whereas the standard procedure involves a “phase factor” $e^{i\infty t}$ which, in a strict mathematical sense, does not exist.

IV. JAYNES-CUMMINGS INTERACTION

Although the above oscillator is formally a “single-mode” one we have seen that the coherent-state averages of Q_t and P_t resemble Fourier decompositions of classical fields.

A single-mode interaction of an oscillator with a two-level system can be solved analytically in the rotating wave approximation (RWA). One knows that, in the standard formalism, oscillations with Rabi frequencies will occur. In our case one expects a *superposition* of different Rabi frequencies and hence a possibility of irreversible spontaneous emission is not excluded.

For a single harmonic oscillator one can repeat standard Heisenberg picture calculations described in detail in [5]. We found it more instructive to begin with the more general case of an arbitrary (or indefinite) number of oscillators. Even at such a general level the problem can be exactly solved.

For a single oscillator the vacuum-picture Hamiltonian is

$$H_{\text{tot}} = \hbar \omega_0 R_3 + \tilde{H} + \hbar \alpha R_2 Q \quad (67)$$

$$= \hbar \omega_0 R_3 + \tilde{H} + \hbar \alpha R_2 \sqrt{\frac{\hbar}{2\Omega m}} (a_\Omega + a_\Omega^\dagger) \quad (68)$$

and its RWA version reads

$$H_{\text{rwa}} = \hbar \omega_0 R_3 + \hbar \Omega a_\Omega^\dagger a_\Omega + i \frac{\hbar \alpha}{2} \sqrt{\frac{\hbar}{2\Omega m}} (R_+ a_\Omega - R_- a_\Omega^\dagger) \quad (69)$$

where $R_k = \sigma_k/2$, $R_\pm = R_1 \pm iR_2$, and α is a real constant. The frequency operator Ω commutes with R_k , a_Ω , a_Ω^\dagger , and H .

The interaction term of the model describing interaction of a two-level system with one oscillator is $\hbar\alpha R_2 Q$. Here Q is the operator representing a configuration-space position of the oscillator. Having a single two-level system interacting with, say, two independent oscillators one expects a term of the form

$$\hbar\alpha_2 R_2 Q_1 + \hbar\alpha_2 R_2 Q_2 \quad (70)$$

where Q_k commute with each other (at equal times). The coupling constant may depend on the number of oscillators. A classical intuition suggests that the greater number of oscillators crowding around the two-level system, the weaker the interaction of a single element of the ensemble. A similar property of coupling constants is found in rigorous approaches to thermodynamic limit in Bose-Einstein condensates [6].

Several different ways of reasoning [1] suggest $\alpha_n = \alpha/\sqrt{n}$ which implies (in the vacuum picture)

$$\underline{H}_{\text{tot}} = \hbar\omega_0 R_3 + \underline{\tilde{H}} + \hbar R_2 \left[\oplus_{\frac{\alpha}{\sqrt{n}}} Q \right] \quad (71)$$

$$= \hbar\omega_0 R_3 + \oplus_1 \hbar\Omega a_\Omega^\dagger a_\Omega + \hbar\alpha R_2 \left[\oplus_{\frac{1}{\sqrt{n}}} \sqrt{\frac{\hbar}{2\Omega m}} (a_\Omega + a_\Omega^\dagger) \right] \quad (72)$$

$$= \hbar\omega_0 R_3 + \sum_{\omega} \oplus_1 \hbar\omega a_\omega^\dagger a_\omega + \hbar\alpha R_2 \sum_{\omega} \sqrt{\frac{\hbar}{2\omega m}} (\underline{a}_\omega + \underline{a}_\omega^\dagger) \quad (73)$$

The RWA Hamiltonian is now

$$\underline{H}_{\text{rwa}} = \hbar\omega_0 R_3 + \underline{\tilde{H}} + i\frac{\hbar\alpha}{2} \sum_{\omega} \sqrt{\frac{\hbar}{2\omega m}} (R_+ \underline{a}_\omega - R_- \underline{a}_\omega^\dagger) \quad (74)$$

$$= \hbar\omega_0 R_3 + \oplus_1 \hbar\Omega a_\Omega^\dagger a_\Omega + i\frac{\hbar\alpha}{2} \left(R_+ \left[\oplus_{\frac{1}{\sqrt{n}}} \sqrt{\frac{\hbar}{2\Omega m}} a_\Omega \right] - R_- \left[\oplus_{\frac{1}{\sqrt{n}}} \sqrt{\frac{\hbar}{2\Omega m}} a_\Omega^\dagger \right] \right) \quad (75)$$

It is clear that our indefinite-frequency oscillators may be regarded as a model of a scalar quantum field interacting with a 2-level atom located at the origin.

In order to compute the evolution of atomic inversion we will proceed in two ways. We will begin with a Dyson expansion and then derive an integral equation which will be solved by means of Laplace transforms. Both methods are instructive and show in different ways links to the standard canonical theory.

We start with

$$\underline{H}_0 = \hbar\omega_0 R_3 + \oplus_1 \hbar\Omega a_{\Omega}^{\dagger} a_{\Omega} \quad (76)$$

$$\underline{H}_1 = i\frac{\hbar\alpha}{2} \left(R_+ \left[\oplus_{\frac{1}{\sqrt{n}}} \sqrt{\frac{\hbar}{2\Omega m}} a_{\Omega} \right] - R_- \left[\oplus_{\frac{1}{\sqrt{n}}} \sqrt{\frac{\hbar}{2\Omega m}} a_{\Omega}^{\dagger} \right] \right) \quad (77)$$

and

$$\underline{H}_{\text{int}}(t) = e^{i\underline{H}_0 t/\hbar} \underline{H}_1 e^{-i\underline{H}_0 t/\hbar} \quad (78)$$

$$= i\frac{\hbar\alpha}{2} \left(e^{i\omega_0 t} R_+ \left[\oplus_{\frac{1}{\sqrt{n}}} \sqrt{\frac{\hbar}{2\Omega m}} e^{-i\Omega t} a_{\Omega} \right] - e^{-i\omega_0 t} R_- \left[\oplus_{\frac{1}{\sqrt{n}}} \sqrt{\frac{\hbar}{2\Omega m}} e^{i\Omega t} a_{\Omega}^{\dagger} \right] \right) \quad (79)$$

$$= i\frac{\hbar\alpha}{2} \sum_{\omega} \left(R_+ \sqrt{\frac{\hbar}{2\omega m}} e^{-i(\omega-\omega_0)t} \underline{a}_{\omega} - R_- \sqrt{\frac{\hbar}{2\omega m}} e^{i(\omega-\omega_0)t} \underline{a}_{\omega}^{\dagger} \right) \quad (80)$$

V. SURVIVAL PROBABILITY

Survival probability is the probability that at $t > 0$ no spontaneous emission occurred.

In the context of our model this means one begins at $t = 0$ with the state

$$\begin{aligned} |\underline{Q}\rangle &= \sum_{\omega} O^{(1)}(\omega) |\omega, 0\rangle |+\rangle \\ &\quad \oplus \sum_{\omega_1, \omega_2} O^{(2)}(\omega_1, \omega_2) |\omega_1, 0\rangle |\omega_2, 0\rangle |+\rangle \\ &\quad \oplus \dots \end{aligned} \quad (81)$$

representing all the oscillators in their ground states and the 2-level system in the excited state. The projector on the subspace consisting of all such states is

$$P = \oplus_{N=1}^{\infty} \sum_{\omega_1 \dots \omega_N} |\omega_1, 0\rangle \langle \omega_1, 0| \otimes \dots \otimes |\omega_N, 0\rangle \langle \omega_N, 0| \otimes |+\rangle \langle +| \quad (82)$$

This should be contrasted with the standard CCR case where there is only one vacuum $|0_{\text{CCR}}\rangle$, the initial state is

$$|O_{\text{CCR}}\rangle = |0_{\text{CCR}}\rangle |+\rangle \quad (83)$$

and the projector is on a one-dimensional subspace,

$$P_{\text{ccr}} = |O_{\text{ccr}}\rangle\langle O_{\text{ccr}}| \quad (84)$$

Denoting, as before, by \underline{H} and $\tilde{\underline{H}}$ the Hamiltonians in Schrödinger and vacuum pictures, respectively, we can consider solutions of

$$i\hbar|\dot{\underline{\psi}}\rangle = \underline{H}|\underline{\psi}\rangle \quad (85)$$

and

$$i\hbar|\dot{\tilde{\underline{\psi}}}\rangle = \tilde{\underline{H}}|\tilde{\underline{\psi}}\rangle \quad (86)$$

which have the same initial condition $|\underline{Q}\rangle$ at $t = 0$. There are now several different but physically natural “survival probabilities”:

$$|\langle \underline{Q} | e^{-i\underline{H}t/\hbar} | \underline{Q} \rangle|^2, \quad (87)$$

$$|\langle \underline{Q} | e^{-i\tilde{\underline{H}}t/\hbar} | \underline{Q} \rangle|^2 \quad (88)$$

and

$$\langle \underline{Q} | e^{i\underline{H}t/\hbar} P e^{-i\underline{H}t/\hbar} | \underline{Q} \rangle = \langle \underline{Q} | e^{i\tilde{\underline{H}}t/\hbar} P e^{-i\tilde{\underline{H}}t/\hbar} | \underline{Q} \rangle, \quad (89)$$

(87) cannot correspond to the probability we want to calculate since even in the absence of interactions (i.e. when $\alpha = 0$) one finds

$$|\langle \underline{Q} | e^{-i\underline{H}t/\hbar} | \underline{Q} \rangle|^2 = |\langle \underline{Q} | e^{-i\Omega t/2} | \underline{Q} \rangle|^2 \neq 1 \quad (90)$$

for $t > 0$. Probabilities (88) and (89) equal 1 for any t if $\alpha = 0$ but are unequal for $\alpha \neq 0$.

In the canonical theory the first of these probabilities is not well defined due to the infinite vacuum energy, but the remaining two are equal since

$$\langle \underline{Q}_{\text{ccr}} | e^{i\tilde{\underline{H}}t/\hbar} P_{\text{ccr}} e^{-i\tilde{\underline{H}}t/\hbar} | \underline{Q}_{\text{ccr}} \rangle = |\langle \underline{Q}_{\text{ccr}} | e^{-i\tilde{\underline{H}}t/\hbar} | \underline{Q}_{\text{ccr}} \rangle|^2, \quad (91)$$

a consequence of one-dimensionality of the vacuum subspace.

In our non-canonical theory the probability which tends (after renormalization of α and for $N \rightarrow \infty$) to the canonical result is (88), as we shall see below.

The vectors

$$|\mathbf{0}_{\omega_1 \dots \omega_N}\rangle = |\omega_1, 0\rangle \dots |\omega_N, 0\rangle |+\rangle \quad (92)$$

$$|\mathbf{1}_{\omega_1 \dots \omega_N}\rangle = |\omega_1, 1\rangle \dots |\omega_N, 0\rangle |-\rangle \quad (93)$$

\vdots

$$|\mathbf{N}_{\omega_1 \dots \omega_N}\rangle = |\omega_1, 0\rangle \dots |\omega_N, 1\rangle |-\rangle \quad (94)$$

span an invariant subspace with respect to the dynamics. The subspace contains at most one oscillator in the first excited state.

Interaction-picture Hamiltonian acts on these vectors as follows

$$\underline{H}_{\text{int}}(t)|\mathbf{0}_{\omega_1 \dots \omega_N}\rangle = -i \frac{1}{\sqrt{N}} \frac{\hbar \alpha}{2} \sum_{k=1}^N \sqrt{\frac{\hbar}{2\omega_k m}} e^{-i(\omega_0 - \omega_k)t} |\mathbf{k}_{\omega_1 \dots \omega_N}\rangle \quad (95)$$

$$\underline{H}_{\text{int}}(t)|\mathbf{1}_{\omega_1 \dots \omega_N}\rangle = i \frac{1}{\sqrt{N}} \frac{\hbar \alpha}{2} \sqrt{\frac{\hbar}{2\omega_1 m}} e^{i(\omega_0 - \omega_1)t} |\mathbf{0}_{\omega_1 \dots \omega_N}\rangle \quad (96)$$

\vdots

$$\underline{H}_{\text{int}}(t)|\mathbf{N}_{\omega_1 \dots \omega_N}\rangle = i \frac{1}{\sqrt{N}} \frac{\hbar \alpha}{2} \sqrt{\frac{\hbar}{2\omega_N m}} e^{i(\omega_0 - \omega_N)t} |\mathbf{0}_{\omega_1 \dots \omega_N}\rangle \quad (97)$$

and can be represented by the matrix

$$H_N(t) = \frac{\hbar}{2} \begin{pmatrix} 0 & -i \frac{\alpha}{\sqrt{N}} \sqrt{\frac{\hbar}{2\omega_1 m}} e^{-i\Delta_{\omega_1} t} & \dots & -i \frac{\alpha}{\sqrt{N}} \sqrt{\frac{\hbar}{2\omega_N m}} e^{-i\Delta_{\omega_N} t} \\ i \frac{\alpha}{\sqrt{N}} \sqrt{\frac{\hbar}{2\omega_1 m}} e^{i\Delta_{\omega_1} t} & 0 & \dots & 0 \\ \vdots & & & \\ i \frac{\alpha}{\sqrt{N}} \sqrt{\frac{\hbar}{2\omega_N m}} e^{i\Delta_{\omega_N} t} & 0 & \dots & 0 \end{pmatrix} \quad (98)$$

where $\Delta_\omega = \omega_0 - \omega$ are the detunings. Eigenvectors of this matrix are analogs of dressed states from the standard formalism. To proceed further define two orthonormal vectors:

$$|1\rangle = \frac{1}{\sqrt{\frac{1}{\omega_1} + \dots + \frac{1}{\omega_N}}} \begin{pmatrix} 0 \\ i\sqrt{\frac{1}{\omega_1}} \\ \vdots \\ i\sqrt{\frac{1}{\omega_N}} \end{pmatrix} \quad (99)$$

$$|0\rangle = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} = |\mathbf{0}_{\omega_1 \dots \omega_N}\rangle \quad (100)$$

and the unitary matrix

$$U_t = \begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & e^{i\Delta\omega_1 t} & \dots & 0 \\ \vdots & & & \\ 0 & 0 & \dots & e^{i\Delta\omega_N t} \end{pmatrix} \quad (101)$$

In this notation

$$H_N(t) = \frac{\alpha\hbar}{2} \sqrt{\frac{\hbar}{2mN} \left(\frac{1}{\omega_1} + \dots + \frac{1}{\omega_N} \right)} \left(|0\rangle\langle 1| U_t^\dagger + U_t |1\rangle\langle 0| \right) \quad (102)$$

In order to compute (88) and (89) it is sufficient to find

$$F(t)_{\omega_1 \dots \omega_N} = \langle 0 | e^{-i\hat{H}t/\hbar} | 0 \rangle \quad (103)$$

Using $\langle 0 | U_t | 1 \rangle = 0$, $\langle 0 | U_t | 0 \rangle = 1$, and denoting

$$f(\tau) = \sum_{k=1}^N \frac{e^{-i\Delta\omega_k \tau}}{\omega_k} \quad (104)$$

one finds

$$F(t)_{\omega_1 \dots \omega_N} = \sum_{n=0}^{\infty} (-1)^n \frac{\alpha^{2n} \hbar^n}{(8mN)^n} \int_0^t dt_1 \int_0^{t_1} dt_2 \dots \int_0^{t_{2n-1}} dt_{2n} f(t_1 - t_2) \dots f(t_{2n-1} - t_{2n}) \quad (105)$$

$$= 1 - \frac{\alpha^2 \hbar}{8mN} \int_0^t dt_1 \int_0^{t_1} dt_2 f(t_1 - t_2) F(t_2)_{\omega_1 \dots \omega_N} \quad (106)$$

Differentiating (106) with respect to t we get

$$\dot{F}(t)_{\omega_1 \dots \omega_N} = -\frac{C}{N} \int_0^t dt_2 f(t - t_2) F(t_2)_{\omega_1 \dots \omega_N}, \quad (107)$$

$C = \frac{\alpha^2 \hbar}{8m}$, whose solution is

$$F(t)_{\omega_1 \dots \omega_N} = \frac{1}{2\pi i} \int_{\Gamma} dz \frac{e^{zt}}{z + \frac{C}{N} \sum_{k=1}^N \frac{1}{\omega_k} \frac{1}{i\Delta_{\omega_k} + z}} \quad (108)$$

where Γ is any contour parallel to the imaginary axis and to the right of all the poles of the integrand.

The poles of the integrand are equal to the eigenvalues of

$$\begin{pmatrix} 0 & -\sqrt{\frac{C}{\omega_1 N}} & \dots & -\sqrt{\frac{C}{\omega_N N}} \\ \sqrt{\frac{C}{\omega_1 N}} & i\Delta_{\omega_1} & 0 & \dots & 0 \\ \vdots & & & & \\ \sqrt{\frac{C}{\omega_N N}} & 0 & \dots & & i\Delta_{\omega_N} \end{pmatrix} \quad (109)$$

and, hence, are purely imaginary.

The probabilities we are interested in are

$$p(t) = \left| \sum_{N=1}^{\infty} \sum_{\omega_1 \dots \omega_N} |O^{(N)}(\omega_1 \dots \omega_N)|^2 F(t)_{\omega_1 \dots \omega_N} \right|^2 \quad (110)$$

for (88), and

$$p(t)' = \sum_{N=1}^{\infty} \sum_{\omega_1 \dots \omega_N} |O^{(N)}(\omega_1 \dots \omega_N)|^2 |F(t)_{\omega_1 \dots \omega_N}|^2 \quad (111)$$

for (89). Of some interest is the fact that a “monochromatic vacuum” with only one frequency, say ω , implies

$$p(t) = \left| \sum_{N=1}^{\infty} |O^{(N)}(\omega \dots \omega)|^2 F(t)_{\omega \dots \omega} \right|^2 = |F(t)_{\omega}|^2 = p(t)' \quad (112)$$

where $|F(t)_{\omega}|^2$ is the solution for $N = 1$. An analogous property will hold for non-CCR quantized field in cavity quantum electrodynamics: Having only one frequency we shall always end up with the standard Rabi oscillations independently of the number of oscillators used to model the field.

To have a better insight into the meaning of the two probabilities it is useful to discuss in more detail the two limiting cases: $N = 1$ and $N \rightarrow \infty$.

VI. $N = 1$ AND RABI OSCILLATIONS

For $N = 1$ one finds the standard solutions

$$F(t)_\omega = \frac{1}{2} \left(1 - \frac{\Delta_\omega}{\sqrt{\Delta_\omega^2 + \frac{\hbar\alpha^2}{2\omega m}}} \right) e^{-\frac{i}{2}(\Delta_\omega + \sqrt{\Delta_\omega^2 + \frac{\hbar\alpha^2}{2\omega m}})t} + \frac{1}{2} \left(1 + \frac{\Delta_\omega}{\sqrt{\Delta_\omega^2 + \frac{\hbar\alpha^2}{2\omega m}}} \right) e^{-\frac{i}{2}(\Delta_\omega - \sqrt{\Delta_\omega^2 + \frac{\hbar\alpha^2}{2\omega m}})t} \quad (113)$$

$$|F(t)_\omega|^2 = 1 - \frac{\hbar\alpha^2}{\hbar\alpha^2 + 2\omega m\Delta_\omega^2} \sin^2 \sqrt{\frac{\hbar\alpha^2}{2\omega m} + \Delta_\omega^2} \frac{t}{2} \quad (114)$$

If vacuum is “flat”, i.e the probabilities are $|O(\omega)|^2 = \text{const}$ for $\omega < \omega_{\text{max}}$ then

$$\lim_{\omega_{\text{max}} \rightarrow \infty} p(t) = \lim_{\omega_{\text{max}} \rightarrow \infty} p(t)' = 1 \quad (115)$$

One can understand this result as follows: For a flat vacuum the emission is dominated by processes far from resonance. However, we shall see in the next section that a correct physical interpretation may require the use of a renormalized coupling constant, and the above interpretation may be premature.

Another interesting case is when one of the frequencies ω equals ω_0 (exact resonance) while the remaining ones are very far from resonance. Then

$$F_\omega(t) \approx \begin{cases} \cos \sqrt{\frac{\hbar}{2\omega_0 m}} \frac{\alpha t}{2}, & \text{for } \omega = \omega_0 \\ 1 & \text{for } \omega \neq \omega_0 \end{cases} \quad (116)$$

Physically this form is much more realistic than the “monochromatic vacuum” discussed at the end of the previous section. Then

$$p(t)' = 1 - |O_{\omega_0}|^2 \sin^2 \sqrt{\frac{\hbar}{2\omega_0 m}} \frac{\alpha t}{2}, \quad (117)$$

$$p(t) = \left| 1 - |O_{\omega_0}|^2 \left(1 - \cos \sqrt{\frac{\hbar}{2\omega_0 m}} \frac{\alpha t}{2} \right) \right|^2 \quad (118)$$

Both expressions coincide if $|O_{\omega_0}|^2 = 1$. The analysis given in the next sections will show that it is $p(t)$ and not $p(t)'$ that agrees with experiment if N is very large. We shall return to this question.

VII. PERTURBATIVE EXPANSION FOR A LARGE NUMBER OF OSCILLATORS

At this point we can estimate to what extent the noncanonical formalism agrees with the usual one. Let us take the truncation $p_{n_{\max}}$ of (110) at the perturbative order $n_{\max} < \infty$. Assume there are exactly N oscillators, $N > n_{\max}$, and the vacuum is of the product form i.e.

$$O^{(N)}(\omega_1 \dots \omega_N) = O_{\omega_1} \dots O_{\omega_N} \quad (119)$$

with $\sum_{\omega} |O_{\omega}|^2 = 1$.

Using (105) we obtain

$$p_{n_{\max}}(t) = \left| \sum_{\omega_1 \dots \omega_N} |O_{\omega_1}|^2 \dots |O_{\omega_N}|^2 \sum_{n=0}^{n_{\max}} (-1)^n \frac{\alpha^{2n} \hbar^n}{(8m)^n} \times \int_0^t dt_1 \dots \int_0^{t_{2n-1}} dt_{2n} \frac{1}{N^n} \sum_{l_1 \dots l_n=1}^N \frac{e^{-i\Delta_{\omega_{l_1}}(t_1-t_2)}}{\omega_{l_1}} \dots \frac{e^{-i\Delta_{\omega_{l_n}}(t_{2n-1}-t_{2n})}}{\omega_{l_n}} \right|^2. \quad (120)$$

The sum under the multiple integral in (120) contains N^n elements which are in a one-to-one relation with points of an n -dimensional cube whose edges have length N and which is embedded in an n -dimensional cubic lattice. Let us denote by N_1 the number of points in this cube whose indices are all different, by N_2 the number of points which have exactly two identical indices, and so on. Using this notation we can write

$$p_{n_{\max}}(t) = \left| \sum_{n=0}^{n_{\max}} \left(\frac{-\alpha^2 \hbar}{8m} \right)^n \int_0^t dt_1 \dots \int_0^{t_{2n-1}} dt_{2n} \left(\frac{N_1}{N^n} \sum_{\omega_1 \dots \omega_n} |O_{\omega_1}|^2 \frac{e^{i\Delta_{\omega_1}(t_2-t_1)}}{\omega_1} \dots |O_{\omega_n}|^2 \frac{e^{i\Delta_{\omega_n}(t_{2n}-t_{2n-1})}}{\omega_n} + \dots + \frac{N_n}{N^n} \sum_{\omega_1} |O_{\omega_1}|^2 \frac{e^{-i\Delta_{\omega_1}(t_1-t_2+\dots+t_{2n-1}-t_{2n})}}{\omega_1^n} \right) \right|^2.$$

It is clear that the last term involves $N_n = N$ and the coefficient $N_n/N^n = 1/N^{n-1} \rightarrow 0$ with $N \rightarrow \infty$ (and $n > 1$). A geometric argument shows that the same holds for any $k > 1$, i.e. $N_k/N^n \rightarrow 0$, while $N_1/N^n \rightarrow 1$. Indeed, the limit $N \rightarrow \infty$ (meaning that one increases the length of the cube's edge while keeping the distance between the lattice points constant)

is geometrically equivalent to the limit where one keeps the length of the edge fixed (say 1) and increases the number of lattice points in $[0, 1]^n$. Then $\lim_{N \rightarrow \infty} N_k/N^n$, $k > 1$, is the probability of finding the point $(x_1, \dots, x_n) \in [0, 1]^n$ whose k coordinates are equal. Sets of such points have n -dimensional measure zero, as sets of geometric dimension at most $n - 1$ (think of probability of hitting a diagonal in a square).

It follows that for a sufficiently large N one can keep only the first term i.e.

$$p_{n_{\max}}(t) \approx \left| \sum_{n=0}^{n_{\max}} (-1)^n \frac{\alpha^{2n} \hbar^n}{(8m)^n} \int_0^t dt_1 \dots \int_0^{t_{2n-1}} dt_{2n} \sum_{\omega_1 \dots \omega_n} |O_{\omega_1}|^2 \frac{e^{i\Delta_{\omega_1}(t_2-t_1)}}{\omega_1} \dots |O_{\omega_n}|^2 \frac{e^{i\Delta_{\omega_n}(t_{2n}-t_{2n-1})}}{\omega_n} \right|^2$$

In the Appendix we show that the right-hand-side of this expression coincides with the perturbative expansion of the survival amplitude in the standard canonical theory, truncated at n_{\max} , whose Hamiltonian is

$$H_{\text{reg}} = \hbar\omega_0 R_3 + \sum_{\omega} \hbar\omega a_{\omega}^{\dagger} a_{\omega} + i \frac{\hbar\alpha}{2} \sum_{\omega} |O_{\omega}| \sqrt{\frac{\hbar}{2\omega m}} (R_+ a_{\omega} - R_- a_{\omega}^{\dagger}) \quad (121)$$

and a_{ω} , a_{ω}^{\dagger} satisfy the CCR algebra with unique vacuum. Square summability of O_{ω} implies $|O_{\omega}| \rightarrow 0$ with $\omega \rightarrow \infty$. It is clear that H_{reg} is the standard RWA Hamiltonian with cut-off functions $|O_{\omega}|$. It is important that the non-CCR formalism introduces the cut-off automatically. A difference with respect to the usual ad hoc regularizations is that the cut-off functions are here summable to unity and, hence, cannot equal 1. Assume that $|O_{\omega}| = A = \text{const}$ until the cut-off region and then decay to 0 in order to guarantee $\sum_{\omega} |O_{\omega}|^2 = 1$. The regularized formulas of the canonical theory agree with the non-canonical ones if one redefines the coupling constant by $\alpha_{\text{exp}} = A\alpha$. The parameter α plays therefore a role of a bare coupling constant [1].

VIII. NONPERTURBATIVE AMPLITUDE FOR A LARGE NUMBER OF OSCILLATORS

The parameters C , N and $\omega_1, \dots, \omega_N$ in (108) are fixed, integration is over any contour localized to the right of all the poles, and the poles are imaginary. It follows that the contour

can be shifted sufficiently far to the right so that the inequality

$$\left| \frac{C}{N} \sum_{k=1}^N \frac{1}{\omega_k} \frac{1}{z} \frac{1}{i\Delta_{\omega_k} + z} \right| < 1 \quad (122)$$

is satisfied and

$$\frac{1}{1 + \frac{C}{N} \sum_{k=1}^N \frac{1}{\omega_k} \frac{1}{z} \frac{1}{i\Delta_{\omega_k} + z}} = \sum_{n=0}^{\infty} \left(-\frac{C}{N} \sum_{k=1}^N \frac{1}{\omega_k} \frac{1}{z} \frac{1}{i\Delta_{\omega_k} + z} \right)^n \quad (123)$$

The amplitude of interest can be thus written as

$$F(t)_{\omega_1 \dots \omega_N} = \frac{1}{2\pi i} \sum_{n=0}^{\infty} (-C)^n \int_{\Gamma} dz \frac{e^{zt}}{z^{n+1}} \frac{1}{N^n} \left(\sum_{k=1}^N \frac{1}{\omega_k} \frac{1}{i\Delta_{\omega_k} + z} \right)^n \quad (124)$$

Repeating the argument from the previous section one can show that for a sufficiently large N

$$\sum_{\omega_1 \dots \omega_N} |O_{\omega_1}|^2 \dots |O_{\omega_N}|^2 F(t)_{\omega_1 \dots \omega_N} \approx \frac{1}{2\pi i} \int_{\Gamma} dz \frac{e^{zt}}{z + C \sum_{\omega} |O_{\omega}|^2 \frac{1}{\omega} \frac{1}{i\Delta_{\omega} + z}} \quad (125)$$

For a large N one expects therefore the same expressions as in the canonical case (cf. the Appendix), but with the coupling constant appropriately regularized by the “vacuum form-factors” $|O_{\omega}|$.

IX. VACUUM ENERGY AND THE COSMOLOGICAL CONSTANT PROBLEM

In standard canonical quantum field theories the vacuum contribution to energy density is [7]

$$\rho_{\text{old}} = \frac{E}{V} = \frac{1}{V} \sum_k \frac{\hbar \omega_k}{2} \approx \frac{\hbar}{4\pi^2 c^3} \int_0^{\omega_{\text{max}}} \omega^3 d\omega = \frac{\hbar}{16\pi^2 c^3} \omega_{\text{max}}^4 \quad (126)$$

where V is the volume and ω_{max} a cut-off frequency. The choice of a concrete value of ω_{max} is a question of taste. One of the typical candidates for ω_{max} is the frequency associated with the Planck scale. The problem is that the resulting estimate on the value of cosmological constant is some 10^{120} times too big when compared with experiments [8–10].

The energy of vacuum in our model of a bosonic quantum field is

$$\langle Q|H|Q\rangle = \frac{\hbar}{2}\langle Q|\Omega|Q\rangle \quad (127)$$

The resulting vacuum energy density is

$$\rho_{\text{new}} = \frac{\langle N \rangle}{V} \sum_k \frac{\hbar \omega_k}{2} |O_\omega|^2 \approx \langle N \rangle \frac{\hbar}{4\pi^2 c^3} \int_0^\infty \omega^3 |O(\omega)|^2 d\omega \quad (128)$$

where $\langle N \rangle$ is the average number of oscillators and we have approximated the discrete probabilities $|O_\omega|^2$ by an appropriate probability density $|O(\omega)|^2$ normalized by

$$\sum_k |O_{\omega_k}|^2 = \frac{V}{2\pi^2 c^3} \int_0^\infty |O(\omega)|^2 \omega^2 d\omega = 1 \quad (129)$$

Assuming for simplicity that $|O(\omega)|^2$ is constant up to some ω_{max} and zero for $\omega > \omega_{\text{max}}$ one finds

$$\rho_{\text{new}} \approx \frac{3}{8} \hbar \omega_{\text{max}} \frac{\langle N \rangle}{V} \quad (130)$$

Assuming further that $\omega_{\text{max}} = 10^a \text{s}^{-1}$ and comparing ρ_{new} with the experimental value $\sim 10^{-47} \text{GeV}^4$ one gets $\langle N \rangle / V \sim 10^{18-a} \text{cm}^{-3}$, which is the average number of oscillators per cubic centimeter in the universe in our toy model of quantum field theory. To compare with atomic data one should not have the cut-off at wavelengths shorter than the Bohr radius, corresponding to $\omega_{\text{max}} \sim 10^{18} \text{s}^{-1}$, which yields roughly one oscillator per cm^3 , or even less. Such a result seems to contradict the idea that the number N of oscillators interacting with atomic electrons is large. More reliable estimates could be derived if one compared the cosmological constant with predictions of non-canonically quantized fields with broken supersymmetry. The analysis shows at least that the “cosmological constant problem” has to be formulated in different terms if non-canonical description of quantum fields is employed.

X. FOCK BUNDLE — THE NON-CANONICAL SPACE OF STATES

The canonical theory has served as a kind of a reference frame for our non-canonical calculations. An important conclusion is that the non-canonical survival probabilities which, for large N , coincide with predictions of the canonical theory have to be computed as if the

set of vacuum states was one dimensional: Survival probabilities, denoted $p(t)$, are averages of the projector $|\underline{Q}\rangle\langle\underline{Q}|$, in exact analogy to the standard CCR formalism.

However, the set of vacuum states is infinite dimensional: There are infinitely many different states annihilated by \underline{a}_Ω . Survival probabilities computed as averages of the projector projecting on the entire subspace of vacuum states, denoted $p(t)'$, typically differ from $p(t)$ and are not expected to agree with experiments.

The ensemble of oscillators forms an object whose properties allow us to regard it as a bosonic (spin-0) quantum field. Non-CCR vacua are Bose-Einstein condensates of the ensemble. States of Bose-Einstein condensates are not unique even if one deals with bosons of the same type (say, sodium atoms) at zero temperature [12].

With each vacuum state one can associate a Fock space of all its excitations. The Fock space is obtained in the standard way by means of the non-CCR algebra of creation and annihilation operators. The set of states has here a structure of a fiber bundle. The base space B is the set of all the vacua: $|\underline{Q}\rangle \in B$ if $\underline{a}_\Omega|\underline{Q}\rangle = 0$. A fiber at $|\underline{Q}\rangle$ is the Hilbert space $\mathcal{H}_{|\underline{Q}\rangle}$ spanned by vectors of the form $\underline{a}_{\omega_1}^\dagger \dots \underline{a}_{\omega_n}^\dagger |\underline{Q}\rangle$, for all $n = 0, 1, 2 \dots$ and all n -tuples $(\omega_1, \dots, \omega_n)$. The perturbative expansion of the vacuum-picture state vector $e^{-i\hat{H}_{\text{rwa}}t/\hbar}|\underline{Q}\rangle$ consists of vectors belonging to $\mathcal{H}_{|\underline{Q}\rangle}$, and the dynamics defines a flow inside of the fiber. Probability $p(t)$ is the survival probability *in the fiber*. The Schrödinger picture may be regarded as a description with “moving fibers”. Transition from Schrödinger’s to vacuum picture removes the part of dynamics in the bundle, namely the motion of a fiber along the trajectory in B caused by zero-energy part of the Hamiltonian. The vacuum picture allows us to work only with the part of the dynamics which is internal to a given fiber.

In a separate paper, when a relativistic non-CCR formalism will be introduced, we shall see that the action of the Poincaré group on the Fock bundle makes vacua covariant and not invariant [13].

XI. DISCUSSION

The discussion of non-canonical quantum optics presented in [1] followed the standard route outlined by Dirac [14]: One starts with classical fields and replaces amplitudes by operators. In the present paper we have reversed the logic of the construction. We have started with a single oscillator and then showed that an ensemble of such oscillators has properties of a non-canonical bosonic quantum field. In order to control physics underlying the formulas we have purposefully restricted the discussion to nonrelativistic oscillators with mass m .

In spite of this restriction the formalism we arrive at has properties strikingly similar to those of radiation fields. The main similarities are found in perturbative formulas describing interactions of such fields with two-level systems. They include Rabi oscillations and spontaneous emission. Having fixed the order of perturbation theory we can always find a finite N which gives predictions equal to the canonical ones within a given precision. To put it differently, for sufficiently large but finite N one expects differences with respect to the canonical theory to be seen in long-time tails.

The main consequence of replacing the parameter ω by an operator is the *automatic* disappearance of both ultraviolet and vacuum infinities. It is therefore not excluded that the missing element of contemporary quantum field theories is that they are not quantized *enough* and are conceptually rooted in the 1925 matrix version [15] of the old-fashioned quantum theory.

The comparison of perturbative calculations in canonical and non-canonical theories suggests that the structure of the space of states appropriate for quantum field theory is not that of the Fock one with unique vacuum, but rather of a vector bundle with all the possible vacua in the role of a base space with Fock-type fibers.

The disappearance of the zero-energy infinity sheds new light on the cosmological constant problem. A natural guess is that an extension of non-canonical methods to superfields with broken supersymmetry is necessary in order to make cosmological predictions more

realistic. Another element which needs consideration is the question of locality of non-CCR-quantized fields. It is quite evident that such fields cannot be local if $|O_\omega| \neq \text{const.}$

These are natural directions for further investigations.

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XII. APPENDIX: COMPARISON WITH THE CANONICAL CASE

We start with the interaction-picture Hamiltonian

$$H_{\text{reg}}(t) = i \frac{\hbar \alpha}{2} \sum_{\omega} |O_{\omega}| \sqrt{\frac{\hbar}{2\omega m}} (R_+ a_{\omega} e^{-i\Delta_{\omega} t} - R_- a_{\omega}^{\dagger} e^{i\Delta_{\omega} t}) \quad (131)$$

and the initial state $|O_{\text{CCR}}\rangle = |0_{\text{CCR}}\rangle|+\rangle$. Creation and annihilation operators satisfy the CCR algebra $[a_{\omega}, a_{\omega'}^{\dagger}] = \delta_{\omega\omega'}$. Since $R_+ = |+\rangle\langle-|$, $R_- = |-\rangle\langle+|$, one has $R_{\pm}^2 = 0$, $\langle+|R_- = 0$, $R_+|+\rangle = 0$, $R_+R_- = |+\rangle\langle+| = (R_+R_-)^n$, $(R_+R_-)^n R_+ = R_+$, $\langle+|(R_+R_-)^n R_{\pm}|+\rangle = 0$.

The perturbative expansion of the survival amplitude (up to an overall phase factor) is

$$F(t) = \langle O_{\text{CCR}} | \sum_{n=0}^{\infty} \left(\frac{1}{i\hbar} \right)^n \int_0^t dt_1 \int_0^{t_1} dt_2 \dots \int_0^{t_{n-1}} dt_n H_{\text{reg}}(t_1) H_{\text{reg}}(t_2) \dots H_{\text{reg}}(t_n) | O_{\text{CCR}} \rangle \quad (132)$$

$$\begin{aligned} &= \sum_{k=0}^{\infty} (-1)^k \frac{\alpha^{2k} \hbar^k}{8^k m^k} \int_0^t dt_1 \int_0^{t_1} dt_2 \dots \int_0^{t_{2k-1}} dt_{2k} \sum_{\omega_1 \dots \omega_{2k}} |O_{\omega_1}| \dots |O_{\omega_{2k}}| \sqrt{\frac{1}{\omega_1}} \dots \sqrt{\frac{1}{\omega_{2k}}} \\ &\quad \times \langle 0_{\text{CCR}} | a_{\omega_1} e^{-i\Delta_{\omega_1} t_1} a_{\omega_2}^{\dagger} e^{i\Delta_{\omega_2} t_2} \dots a_{\omega_{2k-1}} e^{-i\Delta_{\omega_{2k-1}} t_{2k-1}} a_{\omega_{2k}}^{\dagger} e^{i\Delta_{\omega_{2k}} t_{2k}} | 0_{\text{CCR}} \rangle \\ &= \sum_{n=0}^{\infty} (-1)^n \frac{\alpha^{2n} \hbar^n}{(8m)^n} \int_0^t dt_1 \int_0^{t_1} dt_2 \dots \int_0^{t_{2n-1}} dt_{2n} f_O(t_1 - t_2) \dots f_O(t_{2n-1} - t_{2n}) \end{aligned} \quad (133)$$

$$= 1 - \frac{\alpha^2 \hbar}{8m} \int_0^t dt_1 \int_0^{t_1} dt_2 f_O(t_1 - t_2) F(t_2) \quad (134)$$

where

$$f_O(\tau) = \sum_{\omega} |O_{\omega}|^2 \frac{e^{-i\Delta_{\omega}\tau}}{\omega}. \quad (135)$$

The survival amplitude is

$$F(t) = \frac{1}{2\pi i} \int_{\Gamma} dz \frac{e^{zt}}{z + C \sum_{\omega} |O_{\omega}|^2 \frac{1}{\omega_k} \frac{1}{i\Delta_{\omega_k} + z}} \quad (136)$$

Although there are evident similarities between the latter formula and (108), some differences are also very interesting. Of particular importance is the fact that in (108) the frequencies are fixed and the sum is over the number of oscillators. The canonical case (136) involves the sum over all frequencies. Similar differences are between (135) and (104).

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- [2] Let us try to imagine such an oscillator and what it may mean that ω is a quantum number. Let us first recall that for a simple (classical) pendulum in the linear approximation $\omega = \sqrt{g/l}$ where g is the gravitational acceleration and l the length of the pendulum. A simplest “simple quantum pendulum” may be imagined as a two-atom molecule with one of the atoms trapped in some “ceiling” and the other “hanging down”. This is an idealization, of course, but not worse than other idealizations one often makes in quantum mechanics. The role of the length l is played by the distance between centers of mass of the two atoms. Since the two atoms are described by configuration-space wavepackets, they are in superpositions of different center-of-mass localizations. Accordingly, the pendulum is in a superposition of different lengths and, hence, also of ω s. Frequency operator becomes $\Omega = \sqrt{g/|r_1 - r_2|}$ where r_k are center-of-mass operators of the atoms. The limiting values of spectrum of Ω are $\omega = 0$ (if the length is infinite) and $\omega = \infty$ (if the length is zero). For a molecule both conditions can be assumed to occur with zero probability. It follows that the wave function should vanish at $\omega = 0$ and $\omega = \infty$. There exists an analogue of these properties for Fourier transforms of massless fields [3].

- One may criticize the example since such an Ω does not commute with canonical center-of-mass momenta of the atoms. This is true indeed but what it shows is that superpositions of ω s are quite natural if quantum oscillators are considered.

- [3] Ultraviolet infinity is automatically regularized in this formalism by square-integrability of vacuum states since probability of high four-momenta tends to zero. The same property makes vacuum fluctuations finite. It is known that momentum-space wave functions of massless fields are differentiable functions vanishing at the origin $p = 0$ of the light cone [11] a property which is related to the structure of unitary representations of the Poincaré group. One expects that the latter feature will regularize the infrared infinities.

The problem requires relativistic considerations and is beyond the scope of the present paper.

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